RESIT PDE 2020

1) Let $w = |\nabla u|^2 = u_x^2 + u_y^2$

Since $u \in C^1(\bar{\Omega})$, w is continuous on $\partial\Omega$.
Therefore, since $\partial\Omega$ is closed & bounded,
$\exists M$ s.t.

$$w \leq M \text{ on } \partial\Omega$$

An explicit computation gives

$$\Delta u = 2(u_x u_{xxx} + u_{xx}^2 + u_y u_{xxy} + u_{xy}^2 + u_x u_{xyy} + u_{xy}^2 + u_y u_{yyy} + u_{yy}^2)$$

Since u is harmonic:

$$u_x u_{xxx} + u_x u_{xyy} = u_x (u_{xx} + u_{yy})_x = 0$$
$$u_y u_{xyy} + u_y u_{yyy} = u_y (u_{xx} + u_{yy})_y = 0$$

thus

$$\Delta w = 2(u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2) \geq 0 \text{ in } \Omega$$

i.e. u is sub-harmonic in Ω, implying
$|\nabla u|^2 = w \leq M$ in $\bar{\Omega}$.

<u>Remark</u> this can be solved also via FTC or mean value theorem, along w. maximum principle.

(2) (a) As $f(x)$ is odd, all the cosine FS coefficients will be 0. We need only to find the sine Fourier series coefficients

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\,dx$$

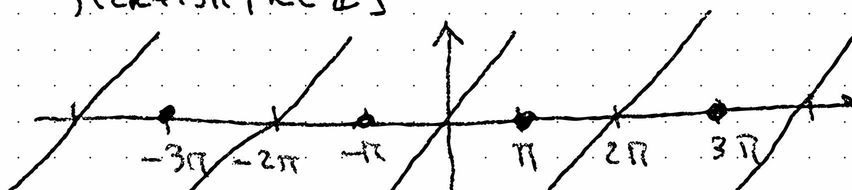
$$= \frac{1}{\pi}\left(-\frac{\cos nx}{n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(-\frac{\cos nx}{n}\right)dx\right)$$

$$= \frac{2}{n\pi}\left(\underbrace{\frac{\sin n\pi}{n}}_{0} - \pi \underbrace{\cos n\pi}_{(-1)^n} \right)$$

$$= \frac{2}{n}(-1)^{n+1}$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx$$

(b) The series converges to the odd periodic extension of $f(x) = x$ in $\mathbb{R} \setminus \{(2k+1)\pi \mid k \in \mathbb{Z}\}$, and $f(x) = 0$ in $\{(2k+1)\pi \mid k \in \mathbb{Z}\}$



(2) (c) $f(x)=x$ is an odd function.

Its periodic extension to $\mathbb{R}$, say f_{ext}, is discontinuous at the points $(2k+1)\pi = x$ as

$$\lim_{x \to (2k+1)\pi^+} f_{ext}(x) = -\pi$$

$$\lim_{x \to (2k+1)\pi^-} f_{ext}(x) = \pi$$

Therefore the Fourier series converges pointwise at $x = \pm\pi$ to $\frac{1}{2}(-\pi+\pi) = 0$.

For $x \in (-\pi, \pi)$, $f_{ext}(x)$ is continuous so the Fourier series converges pointwise to x.

Finally $\|f\|^2 = \int_{-\pi}^{\pi} f(x)^2 dx < +\infty$

$\Rightarrow$ the Fourier sine series converges in L^2 sense.

Uniform convergence is not possible due to jump discontinuities and the Gibbs phenomenon

(3)(a) The free space Green's function is the solution G_0 of

$$-G_0'' + \omega^2 G_0 = \delta(x-\xi) \qquad x, \xi \in \mathbb{R}$$

(b) Assume G_0 to be known, then the superposition principle implies that

$$u(x) = \int_{\mathbb{R}} h(\xi)\, G_0(x;\xi)\, d\xi$$

(c) By applying $\mathcal{F}$ to the equation, we get

$$\hat{u}(k)(k^2+\omega^2) = \hat{h}(k)$$

$$\Rightarrow \quad \hat{u}(k) = \frac{\hat{h}(k)}{k^2+\omega^2} = \hat{h}(k) \cdot \mathcal{F}\left(\sqrt{\frac{\pi}{2}}\, \frac{e^{-\omega|x|}}{\omega}\right)$$

$$= \sqrt{2\pi}\; \hat{h}(k)\, \mathcal{F}\left(\frac{1}{2}\, \frac{e^{-\omega|x|}}{\omega}\right)$$

By the convolution formula

$$u(x) = h * f = \int_{\mathbb{R}} h(\xi)\, \frac{e^{-\omega|x-\xi|}}{2\omega}\, d\xi$$

(4) (a) Solve the eqn for the characteristic curves

$$\frac{dy}{dx} = -3x^2$$

to get $y = -x^3 + C$, where C is the constant of integration.

Solving for C we have

$$C = y + x^3$$

and thus the general solution is of the form

$$u(x,y) = f(C) = f(x^3 + y)$$

where f is an arbitrary function on $\mathbb{R}$.

(b) By the previous point

$$u(0,y) = f(y) = -y^2$$

i.e. $f(z) = -z^2$ and

$$u(x,y) = -(x^3 + y)^2.$$

(5) (a) let $u(r,\theta) = R(r)\,\Theta(\theta)$, substituting in

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$$

we get $R''\,\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0$

Dividing by Θ and R we get

$$r^2 \frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$

which separates into

$$r^2 \frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda,$$

i.e. the equations

$$\begin{cases} r^2 R'' + rR' - \lambda R = 0 \\ \Theta'' + \lambda\Theta = 0 \end{cases}$$

(b) Boundary conditions on Θ imply

$$\Theta(0) = \Theta(\pi) = 0$$

- For $0 < \lambda = k^2$ we have solutions

$$\Theta(\theta) = A\cos(k\theta) + B\sin(k\theta)$$

From $\Theta(0) = A = 0$, $\Theta(\pi) = A\cos(k\pi) + B\sin(k\pi) = 0$

we get $A = 0$, $B\sin(k\pi) = 0$, i.e.

$k_n = n, \quad n = 1, 2, 3, \dots$

- For $\lambda = 0$ we have

$$\Theta(\theta) = A\theta + B$$

So $\Theta(0) = B = 0$ and $\Theta(\pi) = A\pi + B = 0$
which gives the trivial solution
$A = B = 0$, which is not a valid solution.

- For $0 > \lambda = -\gamma^2$ we have

$$\Theta(\theta) = Ae^{\gamma\theta} + Be^{-\gamma\theta}$$

$\Rightarrow$ the boundary conditions imply

$$\Theta(0) = A + B = 0$$

$$\Theta(\pi) = Ae^{\gamma\pi} + Be^{-\gamma\pi} = 0$$

$\Rightarrow \quad B = -A, \; Ae^{-\gamma\pi}(e^{2\gamma\pi} - 1) = 0$

If $\gamma = 0 \Rightarrow \lambda = 0$ but by assumption $\lambda < 0$
If $\gamma \neq 0 \Rightarrow A = B = 0$ which is not a valid solution

In summary, the eigenvalues are

$$\lambda_n = k_n^2 = n^2$$

with eigenfunctions

$$\Theta_n(\theta) = \sin(n\theta)$$

(c) Now that λ is known, the radial solutions are found with the ansatz $R = r^\alpha$
which gives $\alpha = \pm n \Rightarrow R(r) = A_n r^n + B_n r^{-n}$
To be well defined at $r = 0$, we need to

require $B_n = 0$

(d) Putting all the results together, we get

$$u(r,\theta) = \sum_{n \geq 1} A_n r^n \sin(n\theta)$$

For $r = a$ we have

$$u(a,\theta) = \sum_{n \geq 1} A_n a^n \sin(n\theta) = h(\theta)$$

$$\Rightarrow A_n = \frac{2}{\pi a^n} \int_0^\pi h(\theta) \sin(n\theta)\, d\theta$$

In our case

$$\sum_{n \geq 1} A_n a^n \sin(n\theta) = \sin\theta$$

$$\Rightarrow A_1 = 1/a, \quad A_n = 0 \text{ for } n > 1$$

which means

$$u(r,\theta) = \frac{r}{a} \sin\theta.$$